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On the C^1-Norm of the Hermite Interpolation Operator

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1. INTRODUCTION

Consider the interval $I = [-1, 1]$ and the space $C^1(I)$ consisting of the continuously differentiable real-valued functions on I provided with the C^1-norm

$$\|f\|_1 := \max\{\|f\|, \|f'\|\}, \tag{1}$$

where $\|\cdot\|$ denotes the Chebyshev norm on I and f' is the first derivative of $f \in C^1(I)$. We use the Chebyshev nodes of the first kind

$$x_i := \cos y_i := \cos \frac{2i+1}{2n+2}\pi, \qquad 0 \leqslant i \leqslant n, \tag{2}$$

to construct a Hermite interpolation operator

$$H_n\colon C^1(I) \to \Pi_{2n+1}. \tag{3}$$

Pottinger examined the operator norm of H_n induced by (1):

$$\|H_n\|_1 := \max\{\sup_{\substack{f \in C^1(I)\\ \|f\|_1 = 1}} \|H_n f\|, \sup_{\substack{f \in C^1(I)\\ \|f\|_1 = 1}} \|(H_n f)'\|\}$$

$$=: \max\{\|H_n\|_0^*, \|H_n\|_1^*\}.$$

He proved

$$\|H_n\|_1 \geqslant 2n - 4 \qquad [4],$$

$$\|H_n\|_0^* \leqslant 5, \qquad \|H_n\|_1 = O(n)\ (n \to \infty) \qquad [5, 6].$$

In the present paper we give the exact rate of growth of $\|H_n\|_1$; namely, we prove the

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THEOREM. *For the C^1-norm of the Hermite interpolation operator H_n based on the zeros of the Chebyshev polynomials of the first kind, the following inequality holds*:

$$4(n+1)-5+\frac{2}{n+1}\leqslant \|H_n\|_1\leqslant 4(n+1)+\frac{14}{\pi}\ln(n+1)+13.$$

2. SOME ESTIMATES

The Hermite interpolation operator (3) is given by

$$H_n f(x)=\sum_{i=0}^{n} v_i(x)\, l_i^2(x)\, f(x_i)+\sum_{i=0}^{n}(x-x_i)\, l_i^2(x)\, f'(x_i)$$

with

$$v_i(x)=1-\frac{\omega''(x_i)}{\omega'(x_i)}(x-x_i),\qquad 0\leqslant i\leqslant n, \tag{4}$$

and

$$l_i(x)=\frac{\omega(x)}{(x-x_i)\,\omega'(x_i)},\qquad 0\leqslant i\leqslant n,$$

where $\omega(x):=\prod_{i=0}^{n}(x-x_i)$ denotes the nodal polynomial.

Let us first mention a result of Fejér [2]:

LEMMA 1. *For the functions given by* (4), *we have*

$$\sum_{i=0}^{n} v_i(x)\equiv(n+1)^2.$$

From now on we use only the special nodes (2) for which

$$v_i(x)=1-\frac{x_i}{1-x_i^2}(x-x_i)=\frac{1-xx_i}{1-x_i^2}, \tag{5}$$

$$l_i(x)=\frac{(-1)^i}{n+1}\frac{\sin y_i\cos(n+1)\,y}{\cos y-\cos y_i},$$

$0\leqslant i\leqslant n$, where $x=\cos y$, $y\in[0,\pi]$. Using Lemma 1, we have

COROLLARY 1. *For the Chebyshev nodes of the first kind, we have*

$$\sum_{i=0}^{n} \frac{1}{1-x_i} = (n+1)^2, \tag{6}$$

$$\sum_{i=0}^{n} \frac{1}{(1-x_i)^2} = \frac{2}{3}(n+1)^4 + \frac{1}{3}(n+1)^2. \tag{7}$$

Proof. Formula (6) follows immediately from (5) and Lemma 1 for $x = -1$. To prove (7), we take $1 = \pi_0(x) = H_n \pi_0(x)$ and find

$$\begin{aligned} 0 &= (H_n \pi_0)'(1) \\ &= \left\{2 + \frac{1}{(n+1)^2}\right\} \sum_{i=0}^{n} \frac{1}{1-x_i} - \frac{3}{(n+1)^2} \sum_{i=0}^{n} \frac{1}{(1-x_i)^2}. \end{aligned}$$

Using (6), we get the desired result. ∎

LEMMA 2. *The following estimates hold for the functions $A_n, B_n, C_n \in C(I)$ below*:

$$A_n(x) := \sum_{i=0}^{n} \left| \frac{\sin(n+1)\,y}{\sin y} \sin y_i l_i(x) \right|,$$

$$\|A_n\| \leqslant (n+1) + \left\{\frac{2}{\pi} \ln(n+1) + 1\right\}; \tag{8}$$

$$B_n(x) := \sum_{i=0}^{n} \left| \frac{\cos y_i}{\sin^2 y_i} (x - x_i)\, l_i^2(x) \right|,$$

$$\|B_n\| \leqslant \frac{2}{\pi} \ln(n+1) + 1; \tag{9}$$

$$C_n(x) := \sum_{i=0}^{n} \left| \frac{\sin(n+1)\,y}{(n+1)\sin y} \frac{1 - \cos y \cos y_i}{\cos y - \cos y_i} \cos(n+1)\,y \right|,$$

$$\|C_n\| \leqslant (n+1) + 2\left\{\frac{2}{\pi} \ln(n+1) + 1\right\}. \tag{10}$$

Proof. As $A_n(x_i) = 1$, $0 \leqslant i \leqslant n$, let $x \neq x_i$.

Case 1. If $\sin y \leqslant (n+1)^{-1/2}$, then

$$A_n(x) \leqslant (n+1)^{1/2} \sum_{i=0}^{n} |l_i(x)| \leqslant (n+1)^{1/2} \left\{\frac{2}{\pi} \ln(n+1) + 1\right\}.$$

Concerning the estimate of the Lebesgue constant, see Rivlin [7].

Case 2. For $\sin y > (n+1)^{-1/2}$, we get $(x \neq x_i)$

$$\begin{aligned}
A_n(x) &= \frac{1}{n+1} \sum_{i=0}^{n} \frac{|\sin(n+1)\,y|}{\sin y} \sin^2 y_i \left| \frac{\cos(n+1)\,y}{\cos y - \cos y_i} \right| \\
&\leqslant \frac{1}{n+1} \sum_{i=0}^{n} \frac{|\sin(n+1)\,y|}{\sin y} (\sin y_i\, |\sin y_i - \sin y| \\
&\quad + \sin y_i \sin y) \left| \frac{\cos(n+1)\,y}{\cos y - \cos y_i} \right| \\
&\leqslant \sum_{i=0}^{n} \sin y_i \left| \frac{\sin y_i - \sin y}{\cos y - \cos y_i} \right| + \sum_{i=0}^{n} |l_i(x)| \\
&\leqslant \sum_{i=0}^{n} (\sin y_i + \sin y) \left| \frac{\sin y_i - \sin y}{\cos y - \cos y_i} \right| + \sum_{i=0}^{n} |l_i(x)| \\
&= \sum_{i=0}^{n} |\cos y + \cos y_i| + \sum_{i=0}^{n} |l_i(x)| \\
&\leqslant (n+1) + \left\{ \frac{2}{\pi} \ln(n+1) + 1 \right\}.
\end{aligned}$$

As $(n+1)\,|\sin y_i| \geqslant (n+1) \sin y_0 \geqslant (n+1)(2/\pi)\, y_0 = 1$, we obtain (9):

$$B_n(x) \leqslant \frac{1}{(n+1) \sin y_0} \sum_{i=0}^{n} |l_i(x)| \leqslant \frac{2}{\pi} \ln(n+1) + 1.$$

To prove (10), we substitute

$$\begin{aligned}
1 - \cos y \cos y_i &= \sin^2 y + \cos^2 y - \cos y \cos y_i - \sin y \sin y_i \\
&\quad + \sin y \sin y_i \\
&= \sin y(\sin y - \sin y_i) + \cos y(\cos y - \cos y_i) \\
&\quad + \sin y \sin y_i.
\end{aligned}$$

By the triangle inequality

$$\begin{aligned}
C_n(x) &\leqslant \sum_{i=0}^{n} |l_i(x)| + \sum_{i=0}^{n} \frac{|\sin(n+1)\,y|}{(n+1) \sin y} |\cos y \cos(n+1)\,y| \\
&\quad + \sum_{i=0}^{n} \frac{|\sin(n+1)\,y \cos(n+1)\,y|}{n+1} \left| \frac{\sin y - \sin y_i}{\cos y - \cos y_i} \right| \\
&\leqslant \sum_{i=0}^{n} |l_i(x)| + \sum_{i=0}^{n} 1 + \frac{1}{n+1} \sum_{i=0}^{n} \left| \cot \frac{y - y_i}{2} \right|.
\end{aligned}$$

Combining this with the inequality

$$\sum_{i=0}^{n} |l_i(x)| = \frac{1}{n+1} \sum_{i=0}^{n} \cot \frac{y_i}{2} \geqslant \frac{1}{n+1} \sum_{i=0}^{n} \left| \cot \frac{y-y_i}{2} \right|$$

(cf. Ehlich and Zeller [1]) yields the desired result. ∎

3. Proof of the Theorem

We first prove the lower estimate. To this end, we take a sufficiently small $\varepsilon > 0$ and a function $f_\varepsilon \in C^1(I)$ with the properties

$$f_\varepsilon(x_i) = -x_i - i\varepsilon, \qquad 0 \leqslant i \leqslant n;$$

$$f'_\varepsilon(x_i) = 1, \qquad 0 \leqslant i \leqslant n;$$

$$\|f_\varepsilon\|_1 = 1.$$

Then

$$\begin{aligned}
\|H_n\|_1 &\geqslant \|H_n\|_1^* \geqslant |(H_n f_\varepsilon)'(1)| \\
&= \left| \sum_{i=0}^{n} (-x_i - i\varepsilon) \left[\left\{ 2 + \frac{1}{(n+1)^2} \right\} \frac{1}{1-x_i} - \frac{3}{(n+1)^2(1-x_i)^2} \right] \right. \\
&\quad \left. + \sum_{i=0}^{n} 1 \left\{ 2 \frac{\sin^2 y_i}{1-x_i} - \frac{\sin^2 y_i}{(n+1)^2(1-x_i)^2} \right\} \right| \\
&= \left| \varepsilon S - \left\{ 2 + \frac{1}{(n+1)^2} \right\} \sum_{i=0}^{n} \frac{x_i}{1-x_i} + \frac{3}{(n+1)^2} \sum_{i=0}^{n} \frac{x_i}{(1-x_i)^2} \right. \\
&\quad \left. + 2 \sum_{i=0}^{n} (1+x_i) - \frac{1}{(n+1)^2} \sum_{i=0}^{n} \frac{1+x_i}{1-x_i} \right| \\
&= \left| \varepsilon S + \left\{ 2 + \frac{1}{(n+1)^2} \right\} \sum_{i=0}^{n} \left(1 - \frac{1}{1-x_i} \right) \right. \\
&\quad - \frac{3}{(n+1)^2} \left\{ \sum_{i=0}^{n} \frac{1-x_i}{(1-x_i)^2} - \sum_{i=0}^{n} \frac{1}{(1-x_i)^2} \right\} \\
&\quad \left. + 2(n+1) + \frac{1}{(n+1)^2} \sum_{i=0}^{n} \left(1 - \frac{2}{1-x_i} \right) \right|.
\end{aligned}$$

Let $\varepsilon \to 0+$. By Corollary 1

$$\|H_n\|_1 \geqslant \left| \left\{ 2 + \frac{1}{(n+1)^2} \right\} \{(n+1) - (n+1)^2\} \right.$$
$$- \frac{3}{(n+1)^2} \left\{ (n+1)^2 - \frac{2}{3}(n+1)^4 - \frac{1}{3}(n+1)^2 \right\}$$
$$\left. + 2(n+1) + \frac{1}{(n+1)^2} \{(n+1) - 2(n+1)^2\} \right|$$
$$= 4(n+1) - 5 + \frac{2}{n+1}.$$

To prove the upper estimate, we use Pottinger's relation [5, 6]

$$(H_n f)'(x) = \sum_{i=0}^{n} \{v_i'(x)\, l_i^2(x) + 2v_i(x)\, l_i(x)\, l_i'(x)\} \int_x^{x_i} f'(s)\, ds$$
$$+ \sum_{i=0}^{n} \{l_i^2(x) + 2(x - x_i)\, l_i(x)\, l_i'(x)\}\, f'(x_i).$$

Let $f \in C^1(I)$ satisfy $\|f\|_1 = 1$, so that $\|f'\| \leqslant 1$. By Lemma 2,

$$|(H_n f)'(x)| \leqslant \sum_{i=0}^{n} |\{v_i'(x)\, l_i^2(x) + 2v_i(x)\, l_i(x)\, l_i'(x)\}(x - x_i)|$$
$$+ \sum_{i=0}^{n} |l_i^2(x) + 2(x - x_i)\, l_i(x)\, l_i'(x)|$$
$$= \sum_{i=0}^{n} \left| \frac{\cos y_i}{\sin^2 y_i} (x - x_i)\, l_i^2(x) - 2 l_i^2(x) \right.$$
$$\left. + 2 \frac{\sin(n+1)\, y}{(n+1) \sin y} \frac{1 - \cos y \cos y_i}{\cos y - \cos y_i} \cos(n+1)\, y \right|$$
$$+ \sum_{i=0}^{n} \left| -l_i^2(x) + 2 \frac{\sin y_i \sin(n+1)\, y}{\sin y} l_i^2(x) \right|$$
$$\leqslant 2A_n(x) + B_n(x) + 2C_n(x) + 3 \sum_{i=0}^{n} l_i^2(x)$$
$$\leqslant 4(n+1) + 7 \left\{ \frac{2}{\pi} \ln(n+1) + 1 \right\} + 6,$$

since $\sum_{i=0}^{n} l_i^2(x) \leqslant 2$. The proof is complete since $\|H_n\|_0^* \leqslant 5$ (Pottinger [5, 6]). ∎

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